On the connection formula for the first Painleve equation : from the viewpoint of the exact WKB analysis(Painleve Transcendents and Asymptotic Analysis)

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On the connection formula for the first Painlevé equation

— from the viewpoint of the exact WKB analysis —.

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1 Introduction.

In the workshop (held at the Research Institute for Mathematical Sciences, Kyoto University during the period of April 26 – 28, 1995) I gave a series of lectures on the exact WKB analysis of Painlevé transcendents with a large parameter, the subject of my recent research jointly done with T. Kawai (RIMS, Kyoto Univ.) and T. Aoki (Kinki Univ.). Starting with the existence of formal power series solutions λ_J of the J-th Painlevé equation (P_J) ($J =$ I, ..., VI) and the definition of turning points and Stokes curves for them, I explained the following topics mainly in my lectures:

1. local equivalence between λ_{I} and λ_J ($J =$ II, ..., VI),
2. construction of instanton-type formal solutions $\lambda_{J,\alpha,\beta}$ of (P_J),
3. description of the connection formula for λ_{I} in terms of the instanton-type solutions $\lambda_{\mathrm{I},\alpha,\beta}$,
4. local equivalence between $\lambda_{\mathrm{I},\alpha,\beta}$ and $\lambda_{\mathrm{II},\alpha,\beta}$.

The formal power series solutions λ_J and, in particular, their local equivalence are discussed in [KT2] in detail, while the readers should be referred to [AKT3] concerning the topics 2 and 4 in the above list. In this report we concentrate our consideration on the topic 3, i.e., on the connection formula for the first Painlevé equation.

There have been several works about the connection formula for the first Painlevé equation. For example, some Russian mathematicians investigate Painlevé equations by using their relationship with the isomonodromic deformations of linear equations associated with them. (See e.g. [FIK], and references cited there.) In particular, Kapaev has performed asymptotic analysis of the associated linear equation to obtain the leading coefficient of the connection formula for the first Painlevé equation ([K]). On the other hand, employing the so-called multiple-scale analysis, Joshi and Kruskal study the first (and the second) Painlevé equation without resorting to the isomonodromic method ([JK1, JK2]).

Now the purpose of our research is to discuss the Painlevé equation (P_J) in a unified manner through the exact WKB analysis (cf. [Tke]). As a matter of fact, as is shown by the local equivalence between λ_{I} and λ_J, the first Painlevé equation (P_{I}) can be regarded as a canonical equation near a simple turning point of (P_J) from the viewpoint of the exact WKB analysis (just like the Airy equation in the case of linear ordinary differential equations of second order). This, being one of the supporting evidences for the possibility of such a unified treatment of (P_J), implies that explicit analysis of (P_{I}) should be a crucially important step for the determination of the connection formula for (P_J) near a simple turning point. To determine the explicit form of the connection formula for (P_{I}), we shall compute the Stokes multipliers of the linear equation associated with P_{I} by using the exact WKB analysis; to be more specific, we shall employ not only the Voros theory ([V]) but also Kapaev's idea ([K]) to overcome the difficulty of the double turning point whose appearance is a characteristic phenomenon in our theory (cf. [KT1, KT2]). At the present stage the results we have obtained so far are not satisfactory enough (in the sense that the explicit formulas are obtained only for the leading term, not for arbitrarily higher orders), although we hope our method will give us a complete answer in the near future.

By the way, we also note the following fact here; the number of the formal power series solutions λ_J is exactly equal to the number of the local charts, which are constructed explicitly by Takano et al., of 'the space of initial values for (P_J)' (due to Okamoto, cf. [Tka]). As I mentioned at the end of the lectures, our dream (just a dream in the true sense) is that this coincidence is not accidental at all but may be a key property in considering the analytic meaning of instanton-type solutions $\lambda_{J,\alpha,\beta}$ of (P_J). We hope also that this dream will be realized in some future.

In ending this introduction, the author would like to thank Professor T. Kawai and Professor T. Aoki for their many valuable suggestions and the stimulating discussions with them. A part of this work was prepared during the author's stay at the Isaac Newton Institute for Mathematical Sciences supported by the joint research project "Infinite Analysis and Geometry" between the Newton Institute and RIMS, which is sponsored by the Royal Society and the Japan Society for the Promotion of Science.

2 Basic part of the exact WKB analysis for (P_{I}). — A brief review.

First of all, to fix our notations, let us review the basic part of the exact WKB analysis for the first Painlevé equation.

The concrete form of the equation is given by the following:

$$\frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t), \tag{1}$$

which is equivalent to the following Hamiltonian system:

$$\begin{cases} \dfrac{d\lambda}{dt} & = & \eta\dfrac{\partial K_{\mathrm{I}}}{\partial \nu}, \\ \dfrac{d\nu}{dt} & = & -\eta\dfrac{\partial K_{\mathrm{I}}}{\partial \lambda}, \end{cases} \tag{2}$$

where $K_{\mathrm{I}} = \nu^2/2 - (2\lambda^3 + t\lambda)$. Here and in what follows η denotes a large parameter. We also denote the polynomial $6\lambda^2 + t$ by $F_{\mathrm{I}}(\lambda, t)$. As is well-known, the equation (1) (or rather the system (2)) represents the condition for isomonodromic deformations of the following Schrödinger equation in the sense of [JMU] (see [O] and [KT2] also):

$$\left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_{\mathrm{I}}(x, t, \eta)\right)\psi(x, t, \eta) = 0, \tag{3}$$

where the potential Q_{I} is given by

$$Q_{\mathrm{I}} = 4x^3 + 2tx + \nu^2 - (4\lambda^3 + 2t\lambda) - \eta^{-1}\frac{\nu}{x - \lambda} + \eta^{-2}\frac{3}{4(x - \lambda)^2}. \tag{4}$$

Observing this concrete form of the equation, we can easily find that the system (2) (hence the equation (1) also) has the following formal power series solution, which is denoted by $(\lambda_{\mathrm{I}}, \nu_{\mathrm{I}})$, with respect to η^{-1}:

$$(5) \qquad \begin{cases} \lambda_{\mathrm{I}} &= \lambda_0(t) + \eta^{-1}\lambda_1(t) + \eta^{-2}\lambda_2(t) + \cdots, \\ \nu_{\mathrm{I}} &= \nu_0(t) + \eta^{-1}\nu_1(t) + \eta^{-2}\nu_2(t) + \cdots. \end{cases}$$

An important point is that the requirement $(\lambda_{\mathrm{I}}, \nu_{\mathrm{I}})$ should be a solution of (2) forces its leading term (λ_0, ν_0) to satisfy the algebraic equations

$$(6) \qquad F_{\mathrm{I}}(\lambda_0(t), t) = 0 \quad \text{and} \quad \nu_0(t) = 0$$

and also the other terms (λ_j, ν_j) $(j \geq 1)$ to be determined recursively from (λ_0, ν_0). In particular, for λ_{I} the odd order terms all vanish identically and every even order term can be expressed in the following way:

$$(7) \qquad \lambda_{2j} = c_j \left(-\frac{t}{6}\right)^{(1-5j)/2} \qquad (j = 0, 1, 2, \cdots),$$

where $\{c_j\}$ is a series of constants defined by

$$(8) \qquad \begin{cases} c_0 = 1, \quad c_1 = -\dfrac{1}{12^3}, \quad c_2 = -\dfrac{49}{2 \cdot 12^6}, \\ c_j = \dfrac{25}{12^3}(j-1)^2 c_{j-1} - \dfrac{1}{2} \displaystyle\sum_{\substack{k+l=j \\ k,l \geq 2}} c_k c_l \qquad (j \geq 3). \end{cases}$$

(Similarly all of the odd order terms identically vanish for ν_{I}.) As is always the case with singular perturbations, the recursive formula (8) makes λ_{I} diverge. To give an analytic meaning to λ_{I}, we consider its Borel sum, which is the central object of our discussion in this report.

In [KT2] we have defined the turning points and the Stokes curves for formal power series solutions of (P_J). In the case of the equatin (1), i.e., the first Painlevé equation, the origin $t = 0$ is the unique turning point (which is simple) and the Stokes curves become the following:

$$(9) \qquad \{t \in \boldsymbol{C}; \operatorname{Im} \int_0^t \sqrt{\frac{\partial F_{\mathrm{I}}}{\partial \lambda}(\lambda_0(t), t)}\, dt = 0\},$$

namely $\{t \in \boldsymbol{C}; \operatorname{Im}(-t)^{5/4} = 0\}$, five straight lines emanating from the origin. Note that the turning points and the Stokes curves for λ_{I} thus defined have a close relationship with those of the associated Schrödinger equation (3)–(4) (see Proposition 1 in §4.1 below). This relationship (and, in the case of the first Painlevé equation, the concrete analysis of the Borel transform of λ_{I} to be done in the subsequent section as well) explains the reason why the Borel sum of λ_{I} is expected to have a discontinuity on the Stokes curves, as we shall see later.

Our goal in this report is to present, as explicitly as possible, the connection formula describing the discontinuity of the Borel sum of λ_{I} on a Stokes curve. Recalling the definition of the Borel resummation, we may easily guess that the connection formula should be described in terms of the 'instanton-type' solutions introduced in [KT1] first. As a matter of fact, in [AKT3] we have constructed the following formal solutions $\lambda_{\mathrm{I},\alpha,\beta}$ of the equation (1) with two free parameters by employing the multiple-scale analysis:

$$\lambda_{\mathrm{I},\alpha,\beta} = \lambda_0(t) + \eta^{-1/2}\lambda_{1/2}(t,\eta) + \eta^{-1}\lambda_1(t,\eta) + \cdots, \tag{10}$$

where the leading term $\lambda_0(t)$ is the same as that of λ_{I}, i.e., $\lambda_0(t) = s^{1/2}$ $(s = -t/6)$, the sub-leading term $\lambda_{1/2}(t,\eta)$ is given by

$$\lambda_{1/2}(t,\eta) = (12\lambda_0)^{-1/4}\left[\alpha s^{5\alpha\beta/2}e^{\tau} + \beta s^{-5\alpha\beta/2}e^{-\tau}\right] \tag{11}$$

with α and β being arbitrary complex numbers and

$$\tau = \eta\int_0^t \sqrt{\frac{\partial F_{\mathrm{I}}}{\partial\lambda}(\lambda_0(t),t)}\,dt = -\frac{48\sqrt{3}}{5}\left(-\frac{t}{6}\right)^{5/4}\eta, \tag{12}$$

and the higher order terms $\lambda_{j/2}(t,\eta)$ $(j \geq 2)$, which are determined recursively from $\lambda_0(t)$ and $\lambda_{1/2}(t,\eta)$, have the following form:

$$\lambda_{j/2}(t,\eta) = \sum_{k=0}^{j}\lambda_{j-2k}^{(j/2)}(t)e^{(j-2k)\tau}. \tag{13}$$

Remark. Although it is possible to introduce more arbitrary constants into the formal solution $\lambda_{\mathrm{I},\alpha,\beta}$ in determining $\lambda_{(2l-1)/2}(t,\eta)$ for every positive integer l, we assume throught this report that all of them are zero except (α,β) contained in the expression (11) of $\lambda_{1/2}(t,\eta)$. In fact, for our purposes it is sufficient to consider such a simple case due to the distinctive homogeneity (with respect to t and η) that the solution λ_{I} as well as the equation (1) enjoys. For the details concerning these solutions see [AKT3, §1], where the other Painlevé equations are also discussed.

Furthermore, if we assume one of the free parameters, say β, to be zero, then we find that the 'negative instanton-terms' (i.e., the terms $\lambda(t)e^{(j-2k)\tau}$ with $j-2k<0$ in the expression (13) of $\lambda_{j/2}(t,\eta)$) all vanish and the formal solution $\lambda_{\mathrm{I},\alpha,0}$ can be expressed in another way, namely

$$\lambda_{\mathrm{I},\alpha,0} = \lambda^{(0)} + \eta^{-1/2}\lambda^{(1)}e^{\tau} + \eta^{-1}\lambda^{(2)}e^{2\tau} + \cdots, \tag{14}$$

where each $\lambda^{(j)} = \sum_{k\geq 0}\eta^{-k}\lambda_j^{(k+j/2)}(t)$ is a formal power series of η^{-1}. In particular, $\lambda^{(0)}$ coincides with the formal power series solution λ_{I}, $\lambda^{(1)}$ has the form

$$\lambda^{(1)} = \alpha(12\sqrt{s})^{-1/4}\left[1-\frac{15}{4}(12\sqrt{s})^{-5/2}\eta^{-1}+\cdots\right], \tag{15}$$

and the other $\lambda^{(j)}$ ($j\geq 2$) are determined recursively. (A similar expression is possible also for $\lambda_{\mathrm{I},0,\beta}$.) This particular solution $\lambda_{\mathrm{I},\alpha,0}$ (or $\lambda_{\mathrm{I},0,\beta}$), which is nothing but the solution discussed in [KT1, §3], should be an important ingredient of the connection formula for λ_{I}. In the sequel we try to determine the explicit form (or, equivalently, the explicit value of the constant α (or β)) of the solution that appears in the connection formula for λ_{I}.

3 Analytical structure of the Borel transform and the connection formula for λ_{I}.

Before discussing the Stokes multipliers of the linear equation (3)–(4), we investigate some analytical properties of the Borel transform of λ_{I} without applying the isomonodromic method. The content of this section is essentially the same as that of [KT1, §3].

Let $\lambda_{\mathrm{I},B}(t,y)$ denote the Borel transform of λ_{I}. Since the Borel sum of λ_{I} is defined as a Laplace integral

$$\eta\int_0^{\infty} e^{-y\eta}\lambda_{\mathrm{I},B}(t,y)dy,$$

it is reasonable to guess that the Borel sum of λ_{I} may have a discontinuity at a point t_* where a singular point (as a function of y) of $\lambda_{\mathrm{I},B}(t_*,y)$, which we denote by $\phi(t_*)$, lies on the positive real axis and that it should satisfy the following connection formula there:

$$\lambda_{\mathrm{I}} \longrightarrow \lambda_{\mathrm{I}} + e^{-\phi(t)\eta}\widetilde{\lambda} + \cdots . \tag{16}$$

Here $\widetilde{\lambda}$ corresponds to (the Borel sum of) the singular part of $\lambda_{\mathrm{I},B}(t,y)$ at the singular point $y = \phi(t)$. (In (16) we use the same symbol λ_{I} to designate its Borel sum also.) In this manner the analytical structure of the Borel transform $\lambda_{\mathrm{I},B}$ (i.e., the location of its singular points and its singular part at every singular point) completely determines the explicit form of the connection formula for λ_{I}.

On the other hand, we find that the right-hand side of (16) has the same form as that of one parameter instanton-type solutions defined by (14). As a matter of fact, it must be one of the instanton-type solutions due to the "uniqueness" of this kind of solutions ([KT1, Theorem 3.1]) and to the homogeneity of λ_{I}. This implies, under the assumption of the Borel summability of λ_{I}, that the explicit description of instanton-type solutions explained in §2 should be translated into the corresponding information on the analytical structure of $\lambda_{\mathrm{I},B}$ and as a consequence the connection formula for λ_{I} should be determined completely except one constant α which is contained in the expression (14)–(15) of one parameter instanton-type solutions.

From now on let us investigate the analytical structure of $\lambda_{\mathrm{I},B}$ by using its relationship with the explicit description of instanton-type solutions and try to fix the undetermined constant α mentioned above. By the definition the explicit form of $\lambda_{\mathrm{I},B}$ is the following:

$$\lambda_{\mathrm{I},B}(t,y) = \sum_{j=0}^{\infty} \frac{c_j}{(2j)!}\left(-\frac{t}{6}\right)^{(1-5j)/2} y^{2j}, \tag{17}$$

where the coefficient c_j has already been defined by (8). Taking the homogeneity of λ_{I} and $\lambda_{\mathrm{I},B}$ into account, we rewrite (17) as follows:

$$\lambda_{\mathrm{I},B}(t,y) = \left(-\frac{t}{6}\right)^{1/2} \sum_{j=0}^{\infty} d_j z^j, \tag{18}$$

where

$$z = \frac{25}{4\cdot 12^3}\left(-\frac{t}{6}\right)^{-5/2} y^2, \qquad d_j = \left(\frac{4\cdot 12^3}{25}\right)^j \frac{c_j}{(2j)!}. \tag{19}$$

It follows from the recursive formula (8) that the series $\{d_j\}$ satisfies, for example, the following estimate:

$$C4^j\frac{(j-1)!^2}{(2j)!} \le |d_j| \le 4^{j+1}\frac{j!^2}{(2j+2)!} \qquad (j \ge 1) \tag{20}$$

where C is a positive constant. By using this estimate (20) we can readily verify that the sum $\sum d_j z^j$ converges for small z and that its radius of convergence is equal to 1. Furthermore, since d_j is negative for every $j \ge 1$, the point $z = 1$ is a singular point of $\sum d_j z^j$. This immediately implies that for each fixed $t \neq 0$ the Borel transform $\lambda_{\mathrm{I},B}(t,y)$ defines an analytic function near $y = 0$ and has a singularity at $y = \pm\frac{48\sqrt{3}}{5}(-t/6)^{5/4}$ which is denoted by $\pm\phi_{\mathrm{I}}(t)$ hereinafter. The location of this singularity exactly coincides (except the sign) with the one predicted by the explicit form of instanton-type solutions (14) (cf. (12)). As the singular part of $\lambda_{\mathrm{I},B}(t,y)$ at this singular point $y = \phi_{\mathrm{I}}(t)$ corresponds to the coefficient $\eta^{-1/2}\lambda^{(1)}$ of $e^{\tau} = e^{-\phi_{\mathrm{I}}(t)\eta}$ in the expression (14), we can further expect that the function $\sum d_j z^j$ should have the following local expression near the singular point $z = 1$:

$$\sum d_j z^j = (1-z)^{1/2} f(z) + g(z) \tag{21}$$

where f and g are holomorphic functions in a neighborhood of $z = 1$. As a matter of fact, $(1-z)^{1/2}f(z)$ (or, more precisely, the discontinuity of $(-t/6)^{1/2}(1-z)^{1/2}f(z)$) corresponds to the Borel transform of $\eta^{-1/2}\lambda^{(1)}$. (The term $(1-z)^{1/2}$ essentially comes from $\eta^{-1/2}$.) In particular, it follows from this correspondence that the undetermined constant α in the connection formula for λ_{I} is connected with the value of $f(z)$ at $z = 1$ by the relation $\alpha = i\sqrt{5\pi/12}\,\mu$ ($\mu = f(z)|_{z=1}$). (Note that, strictly speaking, this relation is correct except the sign. The sign may depend not only on the choice of various branches but also on which Stokes curves we are considering at.)

Thus our remaining task is to fix the value $\mu = f(1)$. Let us recall that the exponent $j\tau/\eta$ of "instantons" $e^{j\tau}$ in (14) represents the location of singular points of $\lambda_{\mathrm{I},B}(t,y)$. On the other hand, it follows from (12) that the possible value of $j\tau/\eta$ is only an integral multiple of $\phi_{\mathrm{I}}(t)$. This strongly suggests that the point $z = 1$ should be the unique singular point of $\sum d_j z^j$ on its circle of convergence. Let us now assume this fact for the moment. Then it enables us to apply the Darboux method to the analytic function $\sum d_j z^j$. Namely, let $\sum f_k(z-1)^k$ denote the Taylor expansion of $f(z)$ at

$z = 1$ (hence $f_0 = \mu$) and consider the function

$$F_N(z) = \sum_{j=0}^{\infty} d_j z^j - (1-z)^{1/2} \sum_{k=0}^{N-1} f_k (z-1)^k$$

for a positive integer N. It follows from the above assumption and the expression (21) that the holomorphic function $F_N(z)$ in $\{z \in C; |z| < 1\}$ defines an N-times continuously differentiable function on the unit circle $\{z \in C; |z| = 1\}$. Hence the coefficient of the Taylor expansion of $F_N(z)$ at $z = 0$, which is explicitly given by

$$d_j - \sum_{k=0}^{N-1} (-1)^{j+k} f_k \frac{\Gamma(k+3/2)}{j!\Gamma(k+3/2-j)},$$

becomes $O(j^{-M})$ for an arbitrarily given positive number M providing N is sufficiently large, as it is at the same time the Fourier coefficient of the 2π-periodic C^N function $F_N(e^{i\theta})$. Since $(j!\Gamma(k+3/2-j))^{-1} = O(j^{-k-3/2})$ holds for every non-negative integer k, this immediately implies

$$\mu = f_0 = \lim_{j\to\infty} (-1)^j d_j \frac{j!\Gamma(3/2-j)}{\Gamma(3/2)}. \tag{22}$$

Combining (22) and the recursive formula for $\{d_j\}$ (cf. (8) and (19)) together, we thus obtain the following expression of μ:

$$\mu = \lim_{j\to\infty} \mu_j \tag{23}$$

with $\{\mu_j\}$ being defined by the following:

$$\left\{ \begin{aligned} \mu_0 &= 1, \quad \mu_1 = \frac{4}{25}, \quad \mu_2 = \frac{392}{1875}, \\ \mu_j &= \frac{(2j-2)^2}{(2j-1)(2j-3)} \mu_{j-1} \\ &\quad + \frac{1}{2} \sum_{\substack{k+l=j \\ k,l\ge 2}} \frac{(2k-1)!!(2k-3)!!(2l-1)!!(2l-3)!!}{(2j-1)!!(2j-3)!!} \mu_k \mu_l \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j \ge 3) \end{aligned} \right. \tag{24}$$

(where $(2n-1)!! = (2n-1)\cdot(2n-3)\cdot\dots\cdot 1$). A numerical computation of (23)–(24) shows, for example,

$$0.24656174 \le \mu \le 0.24656184. \tag{25}$$

(Note that for the numerical computation of μ it is better to use some error estimate together since the convergence of (23)–(24) is considerably slow.)

The conclusion of this section can be summarized as follows: The study based on the relationship between the analytical structure of the Borel transform $\lambda_{\mathrm{I},B}$ and the explicit description of instanton-type solutions tells us that under several assumptions such as the Borel summability of λ_{I} etc. we have the following connection formula for λ_{I} on each Stokes curve $\{t \in C; \mathrm{Im}(-t)^{5/4} = 0\}$:

$$(26) \qquad \begin{aligned} \lambda_{\mathrm{I}} \;\longrightarrow\; & \lambda_{\mathrm{I},\alpha,0} \\ = \; & \lambda_{\mathrm{I}} + \eta^{-1/2}\lambda^{(1)}e^{-\phi_{\mathrm{I}}(t)\eta} + \eta^{-1}\lambda^{(2)}e^{-2\phi_{\mathrm{I}}(t)\eta} + \cdots, \end{aligned}$$

where

$$(27) \qquad \phi_{\mathrm{I}}(t) = \frac{48\sqrt{3}}{5}s^{5/4} \quad (s = -\frac{t}{6}),$$

$$(28) \qquad \lambda^{(1)} = i\sqrt{\frac{5\pi}{12}}\,\mu\,(12\sqrt{s})^{-1/4}\left[1 - \frac{15}{4}(12\sqrt{s})^{-5/2}\eta^{-1} + \cdots\right],$$

and the other $\lambda^{(j)}$ ($j \geq 2$) are determined recursively. (On some Stokes curves the formula (26) holds with $\lambda_{\mathrm{I},\alpha,0}$ being replaced by $\lambda_{\mathrm{I},0,\beta}$.) The constant μ is the limit of the series $\{\mu_j\}$ defined by (24).

4 Isomonodromic deformations of the associated linear equation and the connection formula for λ_{I}.

In the preceding section we have presented a characterization of the undetermined constant and determined the explicit form of the connection formula for λ_{I} in a quite satisfactory manner. However we have to admit that the exact value of the constant is still not fixed; we can compute it only numerically. In this section we consider our problem by using the relationship of the equation (1) with isomonodromic deformations of the associated linear equation (3)–(4) and try again to fix the constant explicitly.

4.1 Preliminaries.

First of all, let us substitute

$$(29) \qquad \lambda = \lambda_0(t) + \eta^{-1/2}\Lambda, \quad \nu = \eta^{-1/2}\mathcal{N}$$

($\lambda_0(t) = \sqrt{-t/6}$) into the linear equation (3)–(4). After the substitution the potential Q_{I} becomes the following:

$$(30)\qquad \begin{aligned} Q_{\mathrm{I}} &= 4x^3 + 2tx - (4\lambda_0^3 + 2t\lambda_0) + \eta^{-1}(\mathcal{N}^2 - 12\lambda_0\Lambda^2) \\ &\quad - \eta^{-3/2}\left(4\Lambda^3 + \frac{\mathcal{N}}{x-\lambda_0}\right) + \eta^{-2}\frac{-\mathcal{N}\Lambda + 3/4}{(x-\lambda_0)^2} + \cdots. \end{aligned}$$

Now our strategy is as follows: We first compute a pair of two independent Stokes multipliers of the linear equation (3)–(30) by employing the exact WKB analysis due to Voros [V] (see [DDP1] and [AKT2] also). Since a double turning point naturally appears in our situation, we also make use of a local reduction theorem of [AKT3], based on Kapaev's idea ([K]), near the double turning point. As a consequence we will obtain an explicit description of the Stokes multipliers, which actually are infinite series of $\eta^{-1/2}$ with their coefficients being functions of t, Λ and $\mathcal{N}$. Then we solve "isomonodromic equations" (i.e., equations that require these Stokes multipliers to be constant) for Λ and $\mathcal{N}$, regarding t and η as parameters, to seek solutions of the first Painlevé equation (1). If we perform this procedure in two adjacent Stokes regions (i.e., sectorial regions surrounded by Stokes curves for λ_{I}) respectively, the connection formula for λ_{I} will be determined explicitly.

Before starting the computation of Stokes multipliers, let us make some important remarks concerning the WKB-theoretic structure of the equation (3)–(30) here.

First, since the top term Q_0 of Q_{I} is factorized as

$$Q_0 = 4x^3 + 2tx - (4\lambda_0^3 + 2t\lambda_0) = 4(x - \sqrt{s})^2(x + 2\sqrt{s}),$$

where s denotes $-t/6$ as usual, the equation (3)–(30) has a double turning point at $x = \sqrt{s}$. This is really an unpleasant feature; the existence of a double turning point prevents us from employing the Voros theory to compute the Stokes multipliers. We will discuss how to deal with this double turning point in the next subsection.

Second, in connection with this double turning point, the Stokes geometry (i.e., the turning points and the Stokes curves) of the linear equation (3)–(30) is controlled by that of the Painlevé equation (1) in the following manner:

Proposition 1 (i) *At the turning point for λ_{I} (i.e., at $t = 0$) the double turning point $x = \sqrt{s}$ and the other simple turning point $x = -2\sqrt{s}$ of the*

linear equation (3)–(30) collapse to one point.
(ii) *On each Stokes curve (9) for λ_{I} the two turning points $\sqrt{s}$ and $-2\sqrt{s}$ are connected by a Stokes curve of the equation (3)–(30).*

For example, on $\{t \in \boldsymbol{C}; \arg t = 3\pi/5\}$ the following degeneracy of Stokes curves of the linear equation (3)–(30) occurs:

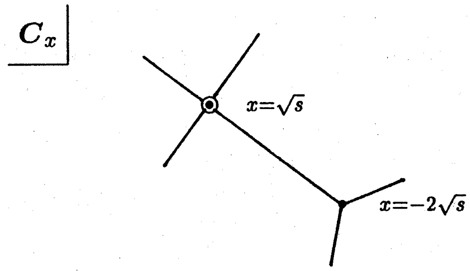

Figure 1: $\arg t = 3\pi/5$

This implies that the configuration of Stokes curves of (3)–(30) is completely different according as $\arg t$ is smaller or greater than $3\pi/5$, as the following two pictures show:

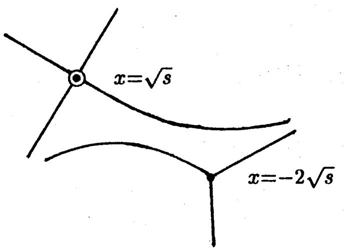

Figure 2: $\arg t < 3\pi/5$

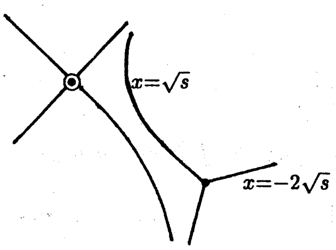

Figure 3: $\arg t > 3\pi/5$

This difference of the configuration of Stokes curves of the linear equation causes the Borel sum of λ_{I} to have a discontinuity on its Stokes curves (in other words, the formal expression of the Borel sum of λ_{I} to become different in an adjacent Stokes region), as we shall see in §4.4.

Remark. These two WKB-theoretically important properties, that is, the existence of a double turning point and the relationship between the two Stokes geometries, are commonly observed for every Painlevé equation (P_J) and its associated linear equation. For the details see [KT2].

4.2 Canonical form near the double turning point.

In order to compute the Stokes multipliers of the equation (3)–(30) we need the connection formula for WKB solutions of (3)–(30) on each Stokes curve emanating from the double turning point $x = \sqrt{s}$. One problem is that the point $x = \sqrt{s}$ is not only a double turning point but also a singular point of (higher order terms of) the potential (30). To overcome this difficulty we make use of a local transformation theorem (Proposition 2 below) near the double turning point, which is based on Kapaev's idea ([K]) and proved in all orders (with respect to $\eta^{-1/2}$) in [AKT3]. In this subsection we study the canonical equation which appears in this local transformation theorem. See also [DDP2] for a general treatment of double turning points.

Now let us first state the local transformation theorem.

Proposition 2 *There exist a neighborhood U of $x = \sqrt{s}$, a formal series*

$$\zeta(x,t,\eta) = \sum_{j=0}^{\infty} \zeta_{j/2}(x,t)\eta^{-j/2}, \tag{31}$$

whose coefficients $\zeta_{j/2}(x,t)$ are holomorphic in $x \in U$ (and also in t), and formal series

$$E(t,\eta) = \sum_{j=0}^{\infty} E_{j/2}(t)\eta^{-j/2}, \tag{32}$$

$$\rho(t,\eta) = \sum_{j=0}^{\infty} \rho_{j/2}(t)\eta^{-j/2}, \tag{33}$$

so that the following four conditions are satisfied:

(34) $\dfrac{\partial\zeta_0}{\partial x}$ *never vanishes,*

(35) $\zeta_0(\sqrt{s},t) = 0,$

(36) $\zeta_{1/2}$ *identically vanishes,*

$$\begin{aligned} Q_{\mathrm{I}}(x,t,\eta) = \left(\frac{\partial\zeta}{\partial x}\right)^2 \Bigg[& 4\zeta(x,t,\eta)^2 + \eta^{-1}E(t,\eta) \\ & + \frac{\eta^{-3/2}\rho(t,\eta)}{\zeta(x,t,\eta) - \zeta(\sqrt{s}+\eta^{-1/2}\Lambda,t,\eta)} \\ & + \frac{3\eta^{-2}}{4(\zeta(x,t,\eta) - \zeta(\sqrt{s}+\eta^{-1/2}\Lambda,t,\eta))^2} \Bigg] \\ & - \frac{1}{2}\eta^{-2}\{\zeta(x,t,\eta);x\}, \end{aligned} \tag{37}$$

where $\{\zeta; x\}$ *denotes the Schwarzian derivative:*

$$\frac{\partial^3\zeta/\partial x^3}{\partial\zeta/\partial x} - \frac{3}{2}\left(\frac{\partial^2\zeta/\partial x^2}{\partial\zeta/\partial x}\right)^2.$$

For the proof of Proposition 2 we refer the reader to [AKT3]. In fact, it can be verified in the same way as [AKT3, Theorem 3.1], although the situation is a little different in that here we assume only the relation (29) while in [AKT3] an instanton-type solution of the system (2) has already been substituted into the potential Q_{I}. Proposition 2 claims that in a neighborhood of the double turning point $x = \sqrt{s}$ the linear equation (3)–(30) should be transformed to the following canonical equation by the transformation (31):

$$\left(-\frac{\partial^2}{\partial\zeta^2} + \eta^2 Q_{\mathrm{can}}(\zeta, t, \eta)\right)\varphi(\zeta, t, \eta) = 0, \tag{38}$$

where

$$Q_{\mathrm{can}} = 4\zeta^2 + \eta^{-1}E(t,\eta) + \frac{\eta^{-3/2}\rho(t,\eta)}{\zeta - \eta^{-1/2}\sigma(t,\eta)} + \frac{3\eta^{-2}}{4(\zeta - \eta^{-1/2}\sigma(t,\eta))^2}. \tag{39}$$

Here we have denoted $\eta^{1/2}\zeta(\sqrt{s}+\eta^{-1/2}\Lambda, t, \eta)$ by $\sigma(t,\eta)$ (which is of order 0 in η by (35)). Note that these formal power series $E(t,\eta)$, $\rho(t,\eta)$ and $\sigma(t,\eta)$, which depend also on Λ and $\mathcal{N}$, are not independent; they satisfy

$$E(t,\eta) = \rho(t,\eta)^2 - 4\sigma(t,\eta)^2 \tag{40}$$

(cf. [AKT3, Remark 3.3]). Furthermore they are determined uniquely in the construction of the transformation. For example, the explicit form of their top order terms is given by

$$E_0 = (3\sqrt{s})^{-1/2}(\mathcal{N}^2 - 12\sqrt{s}\,\Lambda^2), \tag{41}$$

$$\rho_0 = -(3\sqrt{s})^{-1/4}\mathcal{N}, \tag{42}$$

$$\sigma_0 = (3\sqrt{s})^{1/4}\Lambda. \tag{43}$$

In particular, the formal power series $E(t,\eta)$ is related to the original equation (3)–(30) (and hence can be computed easily) by the following relation:

$$E(t,\eta) = 4\,\mathrm{Res}_{x=\sqrt{s}}\, S_{\mathrm{odd}} \tag{44}$$

(cf. [AKT3, (3.33)]), where S_{odd} denotes the generalized odd part (in the sense of [AKT3, Definition 2.1]) of the solution S of the Riccati equation associated with the original equation (3)–(30), that is, $S_{\text{odd}} = (S^{+} - S^{-})/2$ with $S^{\pm}$ being the two formal power series solutions of the Riccati equation.

In the rest of this subsectin we discuss the connection formula for WKB solutions of the canonical equation (38)–(39). Let us denote the solution of the associated Riccati equation by $T = \sum_{j \geq -2} \eta^{-j/2} T_{j/2}(\zeta)$ and its generalized odd part by T_{odd}; for example, $T_{-1} = 2\zeta$ and $T_{-1/2}(\zeta)$ identically vanishes. By the induction it can be readily verified that, for every $j \geq 0$, $T_{j/2}(\zeta)$ contains terms with negative powers in ζ only and

$$\text{(45)} \qquad \operatorname{Res}_{\zeta=0} T_{\text{odd}} = \operatorname{Res}_{\zeta=0} \left(T_{\text{odd}} - \eta T_{-1}\right) = \frac{E(t,\eta)}{4}$$

holds. Thus the following normalized WKB solutions of the equation (38)–(39), which we use as a fundamental system of solutions hereinafter, become well-defined:

$$\text{(46)} \quad \begin{aligned} \varphi_{\pm} \;&=\; \frac{\sqrt{2\eta}}{\sqrt{T_{\text{odd}}}} \zeta^{\pm E/4} \\ &\quad \times \exp \pm \left\{ \eta \int_0^{\zeta} T_{-1} d\zeta + \int_{\infty}^{\zeta} \left(T_{\text{odd}} - \eta T_{-1} - \frac{E}{4\zeta} \right) d\zeta \right\}, \end{aligned}$$

that is,

$$\text{(47)} \qquad \varphi_{\pm} = \exp(\pm \eta \zeta^2)\, \zeta^{-1/2 \pm E/4} \left(1 + O(\eta^{1/2}\zeta)^{-1}\right).$$

Note that the asymptotic behavior (47) uniquely determines the normalization of WKB solutions (in the formal sense). In what follows we consider these WKB solutions $\varphi_{\pm}$ in the Stokes regions (Region I $\sim$ Region III) indicated in Figure 4.

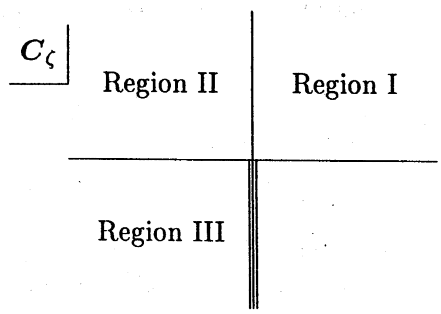

Figure 4

(For our present purposes, i.e., to determine the connection formula for λ_{I}, it is sufficient to consider the problem in these three regions.) We place a cut on the negative imaginary axis and define a branch of $\zeta^{\pm E/4}$ etc. so that $0 < \arg\zeta < 3\pi/2$ may be satisfied.

Now our task is to determine the connection formula for $\varphi_\pm$ between two adjacent regions of Region I $\sim$ Region III. To do so, we transform the canonical equation (38)–(39) into the Weber equation. As a matter of fact, if we first define $\tilde{u}$ and $\tilde{v}$ by

$$(48)\qquad \begin{cases} \tilde{v} &= (\zeta - \eta^{-1/2}\sigma)^{1/2}\varphi, \\ \tilde{u} &= (\zeta - \eta^{-1/2}\sigma)^{-1}\left\{\dfrac{d}{d\zeta}\tilde{v} + \eta^{1/2}\rho\tilde{v}\right\}, \end{cases}$$

and if we, after that, apply the linear transformation

$$(49)\qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \eta^{-1/2} & 2\eta^{1/2} \\ -\eta^{-1/2} & 2\eta^{1/2} \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix},$$

we find

$$(50)\qquad \frac{d}{d\zeta}\begin{pmatrix} u \\ v \end{pmatrix} = \eta \begin{pmatrix} 2\zeta & -\eta^{-1/2}(\rho - 2\sigma) \\ -\eta^{-1/2}(\rho + 2\sigma) & -2\zeta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

In particular, u thus defined satisfies

$$(51)\qquad \frac{d^2u}{d\zeta^2} = \eta^2\left(4\zeta^2 + \eta^{-1}(E+2)\right)u,$$

and the equation (51) can be transformed by the scaling $z = 2\sqrt{\eta}\,\zeta$ of the independent variable into the Weber equation:

$$(52)\qquad \frac{d^2u}{dz^2} + \left(\kappa + \frac{1}{2} - \frac{z^2}{4}\right)u = 0,$$

where $\kappa = -E/4 - 1$. In other words, if for a solution φ of the canonical equation (38)–(39) we define $u(z)$ by

$$(53)\qquad \varphi(\zeta) \longmapsto u(z) \overset{\text{def}}{=} \frac{\sqrt{2}\eta^{1/4}}{\sqrt{z - 2\sigma}}\left[2\frac{d}{dz}\left(\varphi\left(\frac{z}{2\sqrt{\eta}}\right)\right)\right.$$
$$\left. + \left(z + \rho - 2\sigma + \frac{1}{z - 2\sigma}\right)\varphi\left(\frac{z}{2\sqrt{\eta}}\right)\right],$$

then the resulting function $u(z)$ satisfies (52), and conversely

$$\text{(54)}\quad u(z) \longmapsto \varphi(\zeta) \stackrel{\text{def}}{=} \frac{1}{4\eta^{1/2}\sqrt{\zeta - \eta^{-1/2}\sigma}} \times \left[\frac{1}{\rho - 2\sigma}\left(-2\frac{du}{dz} + zu\right) + u\right]\Bigg|_{z=2\sqrt{\eta}\zeta}$$

transforms a solution of the Weber equation (52) to a solution of (38)–(39). Let us denote the former correspondence by $u = \mathcal{G}\varphi$ and the latter by $\varphi = \mathcal{G}^{-1}u$. Making use of this equivalence between the canonical equation and the Weber equation, we compute the connection formula for $\varphi_\pm$ in the following way:

It is well known that the Weber equation (52) has the following formal solution denoted by $\widetilde{D_\kappa}(z)$:

$$\text{(55)}\qquad \widetilde{D_\kappa}(z) = e^{-z^2/4}z^\kappa \sum_{n=0}^{\infty}(-1)^n \frac{\kappa(\kappa-1)\cdots(\kappa-2n+1)}{n!2^n z^{2n}}.$$

Similarly $\widetilde{D_\kappa}(-z)$ and $\widetilde{D_{-\kappa-1}}(\pm iz)$ also satisfy the equation. Now we pick up two of them appropriately and regard them as a fundamental system of solutions of (52) respectively in the Region I′ $\sim$ Region III′ in the complex z-plane as is indicated below:

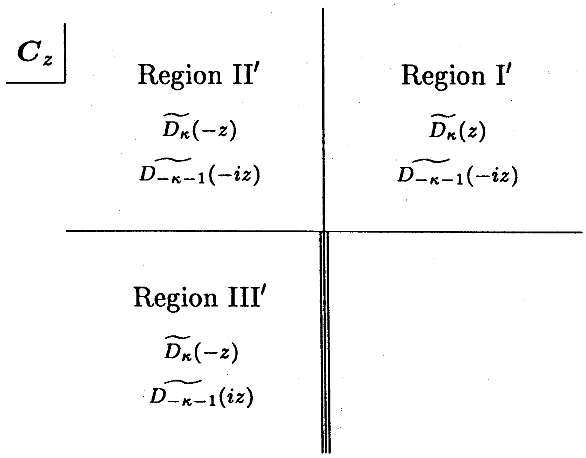

Figure 5

Here we choose a branch of z^κ etc. so that the following may be satisfied in each region:

$$\begin{cases} \text{Region I}': & 0 < \arg z < \pi/2, \quad -\pi/2 < \arg(-iz) < 0, \\ \text{Region II}': & -\pi/2 < \arg(-z) < 0, \quad 0 < \arg(-iz) < \pi/2, \\ \text{Region III}': & 0 < \arg(-z) < \pi/2, \quad -\pi/2 < \arg(iz) < 0. \end{cases}$$

These formal solutions are important in the sense that in every region they correspond to the WKB solutions $\varphi_\pm$ of the equation (38)–(39) by the transformation $\mathcal{G}$ defined above. For example, a straightforward computation shows that in Region I & I′

$$\mathcal{G}^{-1}\widetilde{D_\kappa}(z) = \frac{1}{\rho - 2\sigma}(2\sqrt{\eta})^\kappa e^{-\eta\zeta^2}\zeta^{\kappa+1/2}\left(1 + O((\sqrt{\eta}\zeta)^{-1})\right). \tag{56}$$

This immediately implies

$$\mathcal{G}^{-1}\widetilde{D_\kappa}(z) = \frac{1}{\rho - 2\sigma}(2\sqrt{\eta})^\kappa\varphi_- = \frac{1}{\rho - 2\sigma}(2\sqrt{\eta})^{-E/4-1}\varphi_-. \tag{57}$$

The other correspondences can be computed in a similar manner (though it is necessary to pay some attention to the choice of branches). We write down the complete list of them below:

(58) Region I & I′ :

$$\begin{aligned} \mathcal{G}^{-1}\widetilde{D_\kappa}(z) &= \frac{1}{\rho - 2\sigma}(2\sqrt{\eta})^{-E/4-1}\varphi_-, \\ \mathcal{G}^{-1}\widetilde{D_{-\kappa-1}}(-iz) &= \frac{1}{2}e^{-i\pi E/8}(2\sqrt{\eta})^{E/4-1}\varphi_+, \end{aligned}$$

(59) Region II & II′ :

$$\begin{aligned} \mathcal{G}^{-1}\widetilde{D_\kappa}(-z) &= -\frac{1}{\rho - 2\sigma}e^{i\pi E/4}(2\sqrt{\eta})^{-E/4-1}\varphi_-, \\ \mathcal{G}^{-1}\widetilde{D_{-\kappa-1}}(-iz) &= \frac{1}{2}e^{-i\pi E/8}(2\sqrt{\eta})^{E/4-1}\varphi_+, \end{aligned}$$

(60) Region III & III′ :

$$\begin{aligned} \mathcal{G}^{-1}\widetilde{D_\kappa}(-z) &= -\frac{1}{\rho - 2\sigma}e^{i\pi E/4}(2\sqrt{\eta})^{-E/4-1}\varphi_-, \\ \mathcal{G}^{-1}\widetilde{D_{-\kappa-1}}(iz) &= \frac{1}{2}e^{-3i\pi E/8}(2\sqrt{\eta})^{E/4-1}\varphi_+. \end{aligned}$$

On the other hand, according to the well-known asymptotic formula for the classical Weber function $D_\kappa(z)$ (cf. [BMP, Chapter 8.4]) we can regard the Weber function $D_\kappa(z)$ (resp. $D_\kappa(-z)$, $D_{-\kappa-1}(\pm iz)$) as a Borel sum of the formal solution $\widetilde{D_\kappa}(z)$ (resp. $\widetilde{D_\kappa}(-z)$, $\widetilde{D_{-\kappa-1}}(\pm iz)$) in each region. Furthermore, the well-known formulas (cf. [BMP, Chapter 8.2])

$$D_\kappa(z) = e^{i\pi\kappa}D_\kappa(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\kappa)}e^{i\pi(\kappa+1)/2}D_{-\kappa-1}(-iz) \tag{61}$$

$$D_{-\kappa-1}(-iz) = e^{-i\pi(\kappa+1)}D_{-\kappa-1}(iz) + \frac{\sqrt{2\pi}}{\Gamma(\kappa+1)}e^{-i\pi\kappa/2}D_\kappa(-z) \tag{62}$$

for the Weber function $D_\kappa(z)$ then play a role of the connection formula for the formal solutions $\widetilde{D_\kappa}(z)$ etc. between the two adjacent regions in question. (Note that on the positive imaginary axis $\widetilde{D_{-\kappa-1}}(-iz)$ is a subdominant solution and so is $\widetilde{D_\kappa}(-z)$ on the negative real axis.)

Hence, combining the correspondences (58) $\sim$ (60) and the classical formulas (61)–(62) together, we obtain the connection formula for the WKB solutions $\varphi_\pm$ of the canonical equation (38)–(39). The result is the following:

Proposition 3 *Let $\varphi_\pm^{\mathrm{I}}$ (resp. $\varphi_\pm^{\mathrm{II}}$, $\varphi_\pm^{\mathrm{III}}$) denote the Borel sum of $\varphi_\pm$ in the Region* I *(resp.* II, III*) (cf. Figure 4). Then we have the following relation:*

$$\begin{cases} \varphi_+^{\mathrm{I}} = \varphi_+^{\mathrm{II}} \\ \varphi_-^{\mathrm{I}} = \varphi_-^{\mathrm{II}} + \dfrac{\rho-2\sigma}{2}\dfrac{\sqrt{2\pi}}{\Gamma(E/4+1)}e^{-i\pi E/4}(2\sqrt{\eta})^{E/2}\varphi_+^{\mathrm{II}}, \end{cases} \tag{63}$$

and

$$\begin{cases} \varphi_+^{\mathrm{II}} = \varphi_+^{\mathrm{III}} - \dfrac{2}{\rho-2\sigma}\dfrac{\sqrt{2\pi}}{\Gamma(-E/4)}e^{i\pi(E+1)/2}(2\sqrt{\eta})^{-E/2}\varphi_-^{\mathrm{III}} \\ \varphi_-^{\mathrm{II}} = \varphi_-^{\mathrm{III}}. \end{cases} \tag{64}$$

4.3 Computation of the Stokes multipliers.

Making use of the connection formula of the canonical equation near the double turning point, i.e., Proposition 3 verified in the preceding subsection, and the Voros theory ([V]) as well, we compute the Stokes multipliers of the linear equation associated with the first Painlevé equation in this subsection. For the sake of specification, in other words, in order to fix the configuration of Stokes curves and the branch of various multi-valued functions, we restrict our considerations here to a neighborhood of the Stokes curve $\{t \in C; \arg t = 3\pi/5\}$ for λ_{I}. The other cases can be discussed in the same way.

Now the equation we want to discuss is (3) with the potential Q_{I} given by (30). To compute its Stokes multipliers we need a well-normalized formal solution, which in our situation can be expressed in terms of WKB solutions as follows:

$$(65)\quad \psi_{\pm} = \frac{\sqrt{2\eta}}{\sqrt{S_{\mathrm{odd}}}} \exp \pm \left\{ \eta \int_{-2\sqrt{s}}^{x} S_{-1} dx + \int_{\infty}^{x} (S_{\mathrm{odd}} - \eta S_{-1})\, dx \right\},$$

where $S = \sum_{j \geq -2} \eta^{-j/2} S_{j/2}(x)$ denotes the solution of the Riccati equation associated with (3)–(30) and S_{odd} designates its generalized odd part (cf. [AKT3, Definition 3.1]). For example, the top term $S_{-1}(x)$ is given by

$$S_{-1}(x) = 2(x - \sqrt{s})\sqrt{x + 2\sqrt{s}}$$

($s = -t/6$) whose branch is determined so that $\arg S_{-1}(x) = \pi/5$ may be satisfied on the segment connecting $x = \sqrt{s}$ and $x = -2\sqrt{s}$ (which is nothing but a Stokes curve of the equation (3)–(30)) when $\arg t = 3\pi/5$ (i.e., $\arg s = 8\pi/5$ and $\arg \sqrt{s} = 4\pi/5$). Note that $S_{-1/2}$ identically vanishes and $S_{j/2} = O(x^{-3/2})$ (as $x \to \infty$) holds for every $j \geq 0$. Hence the solutions (65) are well-defined in the formal sense and, since

$$(66)\quad \begin{aligned} \int^{x} S_{-1}(x) dx &= 2\int^{x} (x - \sqrt{s})(x + 2\sqrt{s})^{1/2} dx \\ &= \frac{2}{5}(x - 3\sqrt{s})(x + 2\sqrt{s})^{3/2} \end{aligned}$$

holds (modulo constants depending on t), they have the following asymptotic behavior as $x \to \infty$:

$$(67)\quad \psi_{\pm} = x^{-3/4} \exp\left\{ \pm\eta \left(\frac{2}{5} x^{5/2} + \frac{t}{2} x^{1/2} \right) \right\} \left(1 + O(x^{-1/2}) \right).$$

Our next problem is to determine the connection formula for the normalized WKB solutions $\psi_{\pm}$ on a Stokes curve emanating from the double turning point. We will solve this by using the local transformation between the equation (3)–(30) in question and the canonical equation (38)–(39) provided by Proposition 2. First let $\psi_{\pm}^{J}$ ($J = \mathrm{I}, \mathrm{II}, \mathrm{III}$) denote the Borel sum of the WKB solutions (65) in each region indicated in Figure 6.

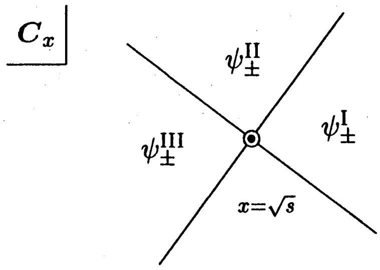

Figure 6: (A local picture near $x = \sqrt{s}$.)

Since the leading term (i.e., the geometric part) $\zeta_0(x,t)$ of the local transformation (31) is defined by

$$\zeta_0(x,t) = \left(2\int_{\sqrt{s}}^{x}(x-\sqrt{s})(x+2\sqrt{s})^{1/2}dx\right)^{1/2}, \tag{68}$$

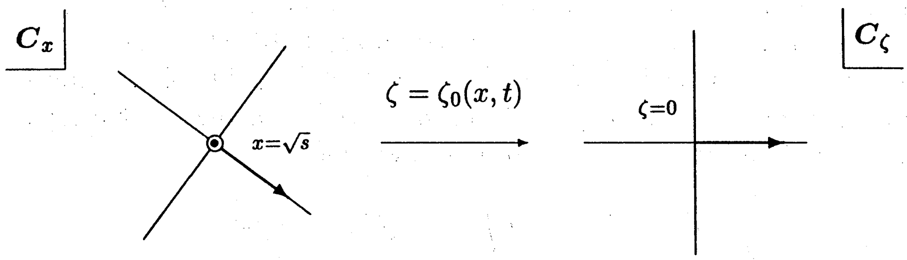

Figure 7: Transformation $\zeta = \zeta_0(x,t)$.

(Note that, as a consequence of our choice of a branch, $\zeta_0(x,t)$ maps the Stokes curve in the x-plane indicated by the arrow to the positive real axis in the ζ-plane.)

the Borel resummed WKB solution $\psi_\pm^J$ (J = I, II, III) should correspond to $\varphi_\pm^J$ respectively under the transformation (31). As a matter of fact, it follows from (37) that $\psi_\pm$ and $\varphi_\pm$ satisfy the following relation in the formal sense:

$$\psi_\pm(x,t,\eta) = C_\pm(t,\eta)\left(\frac{\partial\zeta}{\partial x}\right)^{-1/2}\varphi_\pm(\zeta(x,t,\eta),t,\eta), \tag{69}$$

where $C_\pm(t,\eta)$ are "constants" (i.e., not depending on x) of the following form:

$$C_\pm(t,\eta) = \exp\left(\eta C_{-1}^{(\pm)}(t)\right)\sum_{j=0}^{\infty}\eta^{-j/2}C_{-j/2}^{(\pm)}(t). \tag{70}$$

The relation (69) together with the geometric picture of the transformation $\zeta = \zeta_0(x,t)$ guarantees the correspondence between the Borel sums $\psi_\pm^J$ and $\varphi_\pm^J$ ($J = \mathrm{I}, \mathrm{II}, \mathrm{III}$) (cf. [AKT1]). Furthermore, combining the relation (69) with the connection formula for $\varphi_\pm$ (Proposition 3), we obtain the following connection formula for $\psi_\pm$:

Proposition 4 *The following formulas hold for the solutions* $\{\psi_\pm^J\}$ *(*$J =$ I, II, III*) of the equation (3)-(30):*

$$\left\{\begin{array}{lcl}\psi_+^{\mathrm{I}} & = & \psi_+^{\mathrm{II}} \\ \psi_-^{\mathrm{I}} & = & \psi_-^{\mathrm{II}} + a_{12}\psi_+^{\mathrm{II}}\end{array}\right. \quad \textit{and} \quad \left\{\begin{array}{lcl}\psi_+^{\mathrm{II}} & = & \psi_+^{\mathrm{III}} + a_{23}\psi_-^{\mathrm{III}} \\ \psi_-^{\mathrm{II}} & = & \psi_-^{\mathrm{III}},\end{array}\right. \tag{71}$$

where

$$a_{12} = \frac{C_-}{C_+}\frac{\rho - 2\sigma}{2}\frac{\sqrt{2\pi}}{\Gamma(E/4+1)}e^{-i\pi E/4}(2\sqrt{\eta})^{E/2}, \tag{72}$$

$$\begin{aligned} a_{23} &= -\frac{C_+}{C_-}\frac{2}{\rho - 2\sigma}\frac{\sqrt{2\pi}}{\Gamma(-E/4)}e^{i\pi(E+1)/2}(2\sqrt{\eta})^{-E/2} \\ &= \frac{C_+}{C_-}\frac{\rho + 2\sigma}{2}\frac{\sqrt{2\pi}}{\Gamma(-E/4+1)}e^{i\pi(E+1)/2}(2\sqrt{\eta})^{-E/2}. \end{aligned} \tag{73}$$

(To obtain the last expression of a_{23} we have used the formula $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$ and the relation (40).) As is obvious from Proposition 4, $C_\pm$ should play an important role in our problem. These constants $C_\pm$ are uniquely determined by the transformation (31) and, in principle, can be computed recursively just like ρ and σ, although the actual computation is not easy (especially for higher order terms). Here we only perform the calculation of $C_{-1}^{(\pm)}(t)$ and $C_0^{(\pm)}(t)$, which is sufficient for the present purposes.

Lemma 1

$$\begin{aligned} C_\pm &= \exp\left(\pm\eta\int_{-2\sqrt{s}}^{\sqrt{s}} S_{-1}dx\right)\left(4(3\sqrt{s})^{5/4}\right)^{\mp E_0/4}\left(1 + O(\eta^{-1/2})\right) \\ &= \exp\left(\mp\frac{24\sqrt{3}}{5}s^{5/4}\eta\right)\left(4(3\sqrt{s})^{5/4}\right)^{\mp E_0/4}\left(1 + O(\eta^{-1/2})\right). \end{aligned} \tag{74}$$

Proof. Let $S_{\text{odd},0}$ denote the part of order 0 (with respect to $\eta^{-1/2}$) of S_{odd}. By a straightforward computation we find

$$S_{\text{odd},0} = \frac{\mathcal{N}^2 - 12\lambda_0\Lambda^2}{4(x-\sqrt{s})\sqrt{x+2\sqrt{s}}} = \frac{(3\sqrt{s})^{1/2}E_0}{4(x-\sqrt{s})\sqrt{x+2\sqrt{s}}}.$$

(Here we have used (41).) Since

$$\begin{aligned}(3\sqrt{s})^{1/2}\int^x \frac{dx}{(x-\sqrt{s})\sqrt{x+2\sqrt{s}}} &= \log\frac{\sqrt{x+2\sqrt{s}}-\sqrt{3\sqrt{s}}}{\sqrt{x+2\sqrt{s}}+\sqrt{3\sqrt{s}}} \\ &= \log\frac{x-\sqrt{s}}{\left(\sqrt{x+2\sqrt{s}}+\sqrt{3\sqrt{s}}\right)^2},\end{aligned}$$

we obtain

$$\begin{aligned}(75)\quad \psi_\pm = &\left((x-\sqrt{s})\sqrt{x+2\sqrt{s}}\right)^{-1/2}\exp\left(\pm\eta\int_{-2\sqrt{s}}^x S_{-1}dx\right) \\ &\times (x-\sqrt{s})^{\pm E_0/4}\left(\sqrt{x+2\sqrt{s}}+\sqrt{3\sqrt{s}}\right)^{\mp E_0/2}\left(1+O(\eta^{-1/2})\right).\end{aligned}$$

On the other hand, it follows from (47) and (68) that

$$\begin{aligned}(76)\quad &\left(\frac{\partial\zeta}{\partial x}\right)^{-1/2}\varphi_\pm(\zeta(x,t,\eta),t,\eta) \\ &= \left(\frac{\partial\zeta_0}{\partial x}\right)^{-1/2}\exp\pm\left(\eta\zeta_0^2+2\zeta_0\zeta_1\right)\zeta_0^{-1/2\pm E_0/4}\left(1+O(\eta^{-1/2})\right) \\ &= \left((x-\sqrt{s})\sqrt{x+2\sqrt{s}}\right)^{-1/2}\exp\left(\pm\eta\int_{\sqrt{s}}^x S_{-1}dx\right) \\ &\quad\times\exp(\pm2\zeta_0\zeta_1)\left(\frac{\zeta_0}{x-\sqrt{s}}\right)^{\pm E_0/4}(x-\sqrt{s})^{\pm E_0/4}\left(1+O(\eta^{-1/2})\right).\end{aligned}$$

Let us compare (75) and (76). We have already known that $(\sqrt{x+2\sqrt{s}}+\sqrt{3\sqrt{s}})^{\mp E_0/2}$ and $\exp(\pm2\zeta_0\zeta_1)(\zeta_0/(x-\sqrt{s}))^{\pm E_0/4}$ differ only by a multiplicative constant. (This is an immediate consequence of (69). Using the differential equation for ζ_1, we can confirm this fact also by a straightforward computation.) Hence, if we evaluate them at $x=\sqrt{s}$ (and use $\zeta_0'(\sqrt{s})=(3\sqrt{s})^{1/2}$),

we can easily obtain (74). The last expression of (74) also follows from (66).
Q.E.D.

Now the Stokes multipliers of the equation (3)–(30) can be readily computed. First let us consider the case where $\arg t$ is smaller than $3\pi/5$.

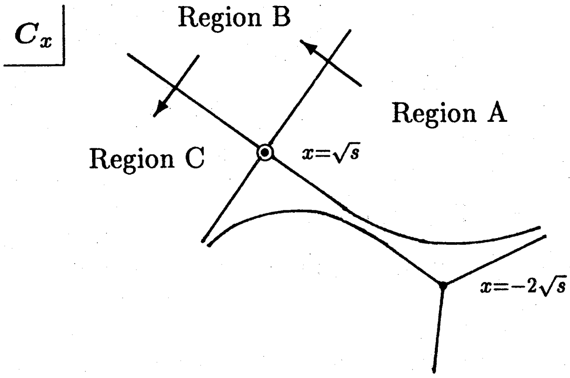

Figure 8: $\arg t < 3\pi/5$

The configuration of Stokes curves (Figure 8) and Proposition 4 imply that the connection matrix for $\psi_\pm$ corresponding to the transfer from Region A to Region B should be the following triangular matrix:

$$\begin{pmatrix} 1 & a_{12} \\ 0 & 1 \end{pmatrix}. \tag{77}$$

Since the Borel sum of $\psi_\pm$ represents a true solution of the equation (3)–(30) with the asymptotic expansion (67) in an appropriate sectorial region (with the center at infinity), the matrix (77) is really one of the Stokes matrices and its off-diagonal component a_{12} gives a Stokes multiplier. We denote this Stokes multiplier by M_{AB} in what follows; namely

$$M_{AB} = a_{12} \qquad (\text{ if } \arg t < 3\pi/5 \text{ }). \tag{78}$$

Similarly, considering the connection matrix corresponding to the transfer from Region B to Region C, which is also a Stokes matrix and whose off-diagonal element gives another Stokes multiplier (independent of the previous one) denoted by M_{BC}, we obtain the following:

$$M_{BC} = a_{23} \qquad (\text{ if } \arg t < 3\pi/5 \text{ }). \tag{79}$$

On the other hand, when $\arg t$ is greater than $3\pi/5$, the configuration of Stokes curves changes as was explained in §4.1 (see Figure 9).

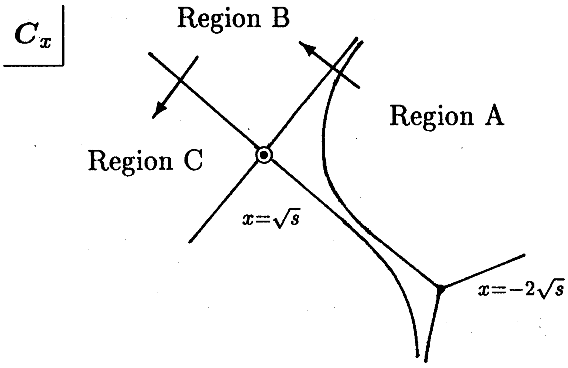

Figure 9: $\arg t > 3\pi/5$

Observing this new configuration, we find that the same result holds for the Stokes multiplier M_{BC}, i.e.,

$$M_{BC} = a_{23} \qquad (\text{ if } \arg t > 3\pi/5 \text{ }). \tag{80}$$

However, for the Stokes multiplier M_{AB} the result differs from the previous one due to the existence of a Stokes curve emanating from the simple turning point $x = -2\sqrt{s}$. In fact, the connection matrix from Region A to Region B is the product of two triangular matrices as follows:

$$\begin{pmatrix} 1 & a_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i \exp 2 \displaystyle\int_{-2\sqrt{s}}^{\infty} (S_{\text{odd}} - \eta S_{-1})\, dx \\ 0 & 1 \end{pmatrix}, \tag{81}$$

where the second matrix is a contribution from the simple turning point. (Its off-diagonal component is the so-called Voros coefficient. Cf. [V].) Note that

$$2 \int_{-2\sqrt{s}}^{\infty} (S_{\text{odd}} - \eta S_{-1})\, dx = \oint_{\text{around } x=\sqrt{s}} (S_{\text{odd}} - \eta S_{-1})\, dx \tag{82}$$

holds (as $S_{\text{odd}} - \eta S_{-1}$ is single-valued on the Riemann sphere with a cut connecting $-2\sqrt{s}$ and ∞, and the point $x = \sqrt{s}$ is the unique singular point of $S_{\text{odd}} - \eta S_{-1}$ on this sphere with the cut). Taking account of the relation (44), we thus obtain the following:

$$M_{AB} = a_{12} + ie^{i\pi E/2} \qquad (\text{ if } \arg t > 3\pi/5 \text{ }). \tag{83}$$

Here let us introduce the following new symbol β, γ and F:

$$\beta = \frac{i}{2}(\rho + 2\sigma) = \frac{i}{2}\left(-\frac{\mathcal{N}}{(3\sqrt{s})^{1/4}} + 2(3\sqrt{s})^{1/4}\Lambda\right) + O(\eta^{-1/2}), \tag{84}$$

(85) $\gamma = \frac{i}{2}(\rho - 2\sigma) = \frac{i}{2}\left(-\frac{\mathcal{N}}{(3\sqrt{s})^{1/4}} - 2(3\sqrt{s})^{1/4}\Lambda\right) + O(\eta^{-1/2}),$

(86) $$F = \log\left\{\frac{C_+}{C_-}e^{i\pi E/4}(2\sqrt{\eta})^{-E/2}\right\}$$
$$= -\frac{48\sqrt{3}}{5}s^{5/4}\eta - \frac{E_0}{4}\log\left(2^6 3^{5/2}s^{5/4}\eta\right) + \frac{i\pi E_0}{4} + O(\eta^{-1/2}).$$

(We have used (42), (43) and Lemma 1 to obtain the explicit form of the leading term of each symbol.) Using these symbols, we can summarize the results obtained in this subsection as follows: The Stokes multipliers M_{AB} and M_{BC} (corresponding to the transfer from Region A to Region B and from Region B to Region C respectively) of the linear equation (3)–(30) has the following expression:

($\arg t < 3\pi/5$)

(87) $$\begin{cases} M_{AB} = -i\gamma e^{(-F)}\dfrac{\sqrt{2\pi}}{\Gamma(E/4+1)} \\ M_{BC} = \beta e^{(i\pi E/4+F)}\dfrac{\sqrt{2\pi}}{\Gamma(-E/4+1)}, \end{cases}$$

($\arg t > 3\pi/5$)

(88) $$\begin{cases} M_{AB} = ie^{i\pi E/2} - i\gamma e^{(-F)}\dfrac{\sqrt{2\pi}}{\Gamma(E/4+1)} \\ M_{BC} = \beta e^{(i\pi E/4+F)}\dfrac{\sqrt{2\pi}}{\Gamma(-E/4+1)}. \end{cases}$$

4.4 Connection formula for λ_{I}.

By the end of the preceding section we have succeeded in computing (the first few terms of) a pair of Stokes multipliers of the linear equation associated with the first Painlevé equation. Making use of this expression of the Stokes multipliers, we explicitly determine the connection formula for λ_{I} in the following way:

Let us consider the following equation ("isomonodromic equation"):

(89) $$M_{AB} = m_1 \quad \text{and} \quad M_{BC} = m_2,$$

where m_1 and m_2 are given complex constants. Note that $\{M_{AB}, M_{BC}\}$ is a pair of independent Stokes multipliers of the linear equation (3)–(30) and every Stokes multiplier of (3)–(30) can be expressed as a rational function of M_{AB} and M_{BC} (cf. [K]). Since the Hamiltonian system (2) represents the condition for isomonodromic deformations of (3)–(30), we can obtain a solution of the system (2) (and, hence, of the first Painlevé equation (1)) by solving the equation (89) for Λ and $\mathcal{N}$ with regarding t and η as parameters for any pair (m_1, m_2) of complex numbers. Keeping this fact in mind, let us first solve the equation (89) by using the explicit formula (87) of M_{AB} and M_{BC} to obtain a solution of the first Painlevé equation when $\arg t < 3\pi/5$. Then we consider the same equation (89) with the other expression (88) of M_{AB} and M_{BC}. Although the expressions of the Stokes multipliers are different for (87) and (88), the analytic continuation of the solution thus obtained in the region $\{t \in \boldsymbol{C}; \arg t < 3\pi/5\}$ should satisfy the equation (89) with the same constants m_1 and m_2 in the region $\{t \in \boldsymbol{C}; \arg t > 3\pi/5\}$ also. Therefore, if we solve the equation (89) with the same constants m_1 and m_2 by using (88), we can obtain an explicit representation of the analytic continuation of the original solution in the region $\{\arg t < 3\pi/5\}$ to the new region $\{\arg t > 3\pi/5\}$. In this way the connection formula across the Stokes curve $\{t \in \boldsymbol{C}; \arg t = 3\pi/5\}$ should be determined explicitly.

As an example, let us investigate the case $m_1 = m_2 = 0$ in greater detail. When $\arg t < 3\pi/5$, the equation $M_{AB} = M_{BC} = 0$ immediately implies

$$\beta = \gamma = 0. \tag{90}$$

Assuming Λ and $\mathcal{N}$ have the expansion of the form

$$\begin{cases} \Lambda &= \Lambda_0 + \eta^{-1/2}\Lambda_{1/2} + \cdots \\ \mathcal{N} &= \mathcal{N}_0 + \eta^{-1/2}\mathcal{N}_{1/2} + \cdots, \end{cases} \tag{91}$$

we can easily solve the equation (90). The result is $\Lambda_0 = \mathcal{N}_0 = 0$ (cf. (84) and (85)) and the corresponding solution of the first Painlevé equation (1) becomes the formal power series solution λ_{I}. On the other hand, when $\arg t > 3\pi/5$, we still have $\beta = 0$ and hence

$$E = \rho^2 - 4\sigma^2 = -4\beta\gamma = 0 \tag{92}$$

follows from (40), (84) and (85). However, $\gamma = 0$ does not hold. Instead we have to solve the following system of equations:

$$\beta = 0 \qquad \text{and} \qquad \sqrt{2\pi}e^{(-F)}\gamma = 1, \tag{93}$$

or, more explicitly,

$$\beta = 0 \quad \text{and} \quad \gamma = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{48\sqrt{3}}{5} s^{5/4}\eta\right)\left(1 + O(\eta^{-1/2})\right). \tag{94}$$

If we assume the expansion (91) again and further regard $\exp(-48\sqrt{3}s^{5/4}\eta/5)$ as a quantity of order 0 in η (just like in the construction of the multiple-scale solution (10) discussed in [AKT3]), the equation (94) implies

$$\Lambda_0 = i\frac{1}{2\sqrt{\pi}}\left(12\sqrt{s}\right)^{-1/4} \exp\left(-\frac{48\sqrt{3}}{5}s^{5/4}\eta\right), \tag{95}$$

that is,

$$\lambda = \sqrt{s} + \eta^{-1/2} i\frac{1}{2\sqrt{\pi}}\left(12\sqrt{s}\right)^{-1/4} \exp\left(-\frac{48\sqrt{3}}{5}s^{5/4}\eta\right) + \cdots. \tag{96}$$

This expression (96) of λ should be the analytic continuation of λ_{I} in the region $\{\arg t > 3\pi/5\}$.

Thus we have explicitly determined the connection formula for λ_{I} across the Stokes curve $\{t \in \boldsymbol{C}; \arg t = 3\pi/5\}$ as follows:

$$\begin{aligned} \lambda_{\mathrm{I}} &\longrightarrow \lambda_{\mathrm{I},i/2\sqrt{\pi},0} \\ &= \sqrt{s} + \eta^{-1/2} i\frac{1}{2\sqrt{\pi}}\left(12\sqrt{s}\right)^{-1/4} \exp\left(-\frac{48\sqrt{3}}{5}s^{5/4}\eta\right) + \cdots. \end{aligned} \tag{97}$$

In particular, by the comparison of (97) and (26) $\sim$ (28) we have

$$\mu = \sqrt{\frac{3}{5}\frac{1}{\pi}} = 0.24656177\cdots, \tag{98}$$

which coincides with the previous numerical result (25) very accurately.

This is our determination of the connection formula for λ_{I} by using the isomonodromic method. In my opinion the isomonodromic method is more powerful than the argument performed in the previous section (§3). (For example, it enables us to determine the undetermined constant α (or, more specifically, μ in the formula (28)) in a completely explicit manner.) We hope that this method will determine the connection formula not only for the formal power series solution λ_{I} but also for any multiple-scale solution $\lambda_{\mathrm{I},\alpha,\beta}$ with two free parameters. (Here a free parameter means an arbitrary function or formal series of $\eta^{-1/2}$ that is free from t. See Remark in §2 also.) However, to realize this expectation we have to compute the connection formula for WKB solutions of the associated linear equation near the double turning point $x = \sqrt{s}$, in particular the constants $C_\pm$ in (69)–(70), up to an arbitrarily higher order, and at the present stage this is still an open problem.

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